

Schematic Harder-Narasimhan stratification for families of principal bundles in higher dimensions

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Abstract

Let G be a connected split reductive group over a field k of characteristic zero. Let $X \rightarrow S$ be a smooth projective morphism of k -schemes, with geometrically connected fibers. We formulate a natural definition of a relative canonical reduction, under which principal G -bundles of any given Harder-Narasimhan type τ on fibers of X/S form an Artin algebraic stack $Bun_{X/S}^\tau(G)$ over S , and as τ varies, these stacks define a stratification of the stack $Bun_{X/S}(G)$ by locally closed substacks. This result extends to principal bundles in higher dimensions the earlier such result for principal bundles on families of curves. The result is new even for vector bundles, that is, for $G = GL_{n,k}$.

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1 Introduction

Let G be a connected split reductive algebraic group over a field k of characteristic zero. Let there be chosen a split maximal torus and a Borel containing it, and let \overline{C} denote the corresponding closed positive Weyl chamber. If $(X, \mathcal{O}_X(1))$ is a projective variety over an extension field K/k together with a very ample line bundle, and if E is a principal G -bundle on X , then recall that the Harder-Narasimhan type of E is an element $\text{HN}(E) \in \overline{C}$. It is defined as the type of the canonical reduction of E , which is a particular rational reduction of the structure group of E , from G to a standard parabolic P . Here, a rational reduction is a reduction defined on a big open subscheme of X , that is, an open subscheme whose complement is of codimension ≥ 2 .

We now move to the relative set-up. Let $X \rightarrow S$ be a smooth projective morphism with geometrically connected fibers, where S is a noetherian scheme over k , with a given relatively very ample line bundle. A family of principal G -bundles on X/S means a principal G -bundle E on X , so that each restriction $E_s = E|_{X_s}$ is a principal G -bundle on the smooth projective variety X_s which is the fiber of X over $s \in S$. The individual Harder-Narasimhan types $\text{HN}(E_s) \in \overline{C}$ together define a function $S \rightarrow \overline{C}$. We prove that (see Proposition 7.4 assertion (1)) this function is upper semi-continuous w.r.t. the usual partial ordering on \overline{C} . This implies that for any $\tau \in \overline{C}$, the subset $|S|^{\leq \tau}(E)$ which consists of all $s \in S$ such that $\text{HN}(E|_{X_s}) \leq \tau$ is open in S , the subset $|S|^\tau(E)$ which consists of all $s \in S$ such that $\text{HN}(E|_{X_s}) = \tau$ is closed in $|S|^{\leq \tau}(E)$, and the closure of $|S|^\tau(E)$ in $|S|$ is contained in $\bigcup_{\gamma \geq \tau} |S|^\gamma(E)$. Hence for each τ the Artin stack $Bun_{X/S}(G)$ of principal G -bundles over X/S has an

open substack $Bun_{X/S}^{\leq \tau}(G)$ which to any S -scheme T associates the groupoid whose objects are all principal G -bundles E on X_T such that $\mathrm{HN}(E_t) \leq \tau$ at all $t \in T$.

If we are interested not just in the discrete invariant $\mathrm{HN}(E_s)$ but in the behaviour of the canonical reductions of E_s in a family, then we need to have a good, workable definition of a relative rational reduction of structure group of such an E , from G to a standard parabolic P . On the one hand, in the special case where $S = \mathrm{Spec} K$ for an extension field K/k , the definition of a relative rational reduction to P should amount just to a reduction to P over a big open subscheme $U \subset X$. On the other hand, the definition should have a modicum of \mathcal{O} -coherence and flatness built into it which would allow us to deploy the theory of flatness and base change for coherent sheaves and their cohomologies.

We now describe our candidate for such a definition. To begin with, to each standard parabolic P , we associate the irreducible linear G -representation $V_P = \Gamma(G/P, \omega_{G/P}^{-1})^\vee$. As the anti-canonical line bundle $\omega_{G/P}^{-1}$ is very ample, it defines a G -equivariant embedding $G/P \hookrightarrow \mathbf{P}(V_P)$, which denotes the projective space of lines in V_P .

Let $X \rightarrow S$ be a smooth projective morphism with geometrically connected fibers, where S is a noetherian scheme over k , with a given relatively very ample line bundle $\mathcal{O}_{X/S}(1)$ on X . Let E be a principal G -bundle on X . Let $E(V_P)$ denote the vector bundle associated to E by the G -representation V_P . We define a **relative rational reduction** of structure group of E from G to P to be a pair (L, f) , where L is a line bundle on X and $f : L \rightarrow E(V_P)$ is an injective \mathcal{O}_X -linear homomorphism of sheaves, such that

- (i) the open subscheme $U = \{x \in X \mid \mathrm{rank}(f_x) = 1\} \subset X$ is relatively big over S , that is, for each $s \in S$ the fiber U_s has complementary codimension ≥ 2 in the fiber X_s , and
- (ii) the section $U \rightarrow \mathbf{P}(E(V_P))$ defined by f factors via the natural closed embedding $E/P \hookrightarrow \mathbf{P}(E(V_P))$.

Note that a section $\sigma : U \rightarrow \mathbf{P}(E(V_P))$ is the same as a line subbundle $f' : L' \hookrightarrow E(V_P)|_U$, which is the pullback by σ of the tautological line subbundle $\mathcal{O}(-1) \hookrightarrow E(V_P)_{\mathbf{P}(E(V_P))}$. By Proposition 2.2, giving the extra data (L, f) simply amounts to imposing the requirement that L' should admit a prolongation to a line bundle L on X . If it exists, such a prolongation L will be unique, and in that case there will exist a unique map $f : L \rightarrow E(V_P)$ which prolongs f' .

In the special case where $S = \mathrm{Spec} K$ for a field K , the above definition is equivalent to the usual definition (this is the Proposition 3.4), as one would require of any such generalization. Finally, we define a **relative canonical reduction** for E on X over S to be a relative rational reduction (L, f) of the structure group of E from G to a standard parabolic P , such that for each $s \in S$, the restriction $(L|_{X_s}, f|_{X_s})$ is a canonical reduction of $E_s = E|_{X_s}$.

For any type $\tau \in \overline{C}$, the above definition allows us to define an S -groupoid $Bun_{X/S}^\tau(G)$, which attaches to any S -scheme T the category $Bun_{X/S}^\tau(G)(T)$ whose objects are

(E, L, f) where E is a principal G -bundle on X_T and (L, f) is a relative canonical reduction for E of constant type τ . We prove that (see Proposition 7.4 assertion (2)) if a relative canonical reduction exists, then it is unique up to a unique isomorphism, that is, given any two such reductions (L, f) and (L', f') , there exists a unique isomorphism $\phi : L \rightarrow L'$ such that $f = f' \circ \phi$. The following is our main result.

Theorem 1.1 *For any $\tau \in \overline{C}$, the forgetful 1-morphism $\text{Bun}_{X/S}^\tau(G) \rightarrow \text{Bun}_{X/S}(G)$ is a locally closed embedding of Artin stacks. As τ varies over \overline{C} , this defines a stratification of $\text{Bun}_{X/S}(G)$ by the locally closed substacks $\text{Bun}_{X/S}^\tau(G)$ with respect to the standard partial order on \overline{C} , that is, each $\text{Bun}_{X/S}^\tau(G)$ is a closed substack of the open substack $\text{Bun}_{X/S}^{\leq \tau}(G)$ of $\text{Bun}_{X/S}(G)$.*

The reader is cautioned that while the closure of $\text{Bun}_{X/S}^\tau(G)$ in $\text{Bun}_{X/S}(G)$ will be contained in $\cup_{\gamma \geq \tau} \text{Bun}_{X/S}^\gamma(G)$, the inclusion will in general be proper.

The above result has the following formulation in elementary terms.

Theorem 1.2 *Let E be a principal G -bundle on a smooth projective family of varieties X/S . Then for each $\tau \in \overline{C}$, there exists a locally closed subscheme $S^\tau(E) \subset S$ with the following universal property: A morphism $T \rightarrow S$ factors via the inclusion $S^\tau(E) \hookrightarrow S$ if and only if the pullback E_T of E to X_T admits a relative canonical reduction of constant type τ over T . As τ varies over \overline{C} , this defines a stratification of S by the locally closed subschemes $S^\tau(E)$ with respect to the standard partial order on \overline{C} .*

Note that in particular, the Theorem 1.2 asserts that there exists a relative canonical reduction in our sense over each $S^\tau(E)$. This is stronger than just the conclusion that there exists a relatively big open subscheme in $X \times_S S^\tau(E)$ over which we have a parabolic reduction which restricts to the canonical reduction on each fiber X_s for $s \in S^\tau(E)$.

When S is reduced and $\text{HN}(E_s)$ is constant over S , the above theorem gives the existence of a relative canonical reduction over S , in the strong sense of our above new definition of such a relative reduction. By forgetting the extra data involved, this has the following immediate consequence, which – as the reader may note – makes no reference at all to our new definition of a relative canonical reduction.

Corollary 1.3 *Let E be a principal G -bundle on a smooth projective family of varieties $(X, \mathcal{O}_{X/S}(1))$ on a noetherian k -scheme S . If S is reduced and if the Harder-Narasimhan type $\text{HN}(E_s)$ is constant over S , then there exists a relative canonical reduction (L, f) of E over S . In particular, there exists a parabolic reduction $\sigma : U \rightarrow E/P$ of E that is defined on a relatively big open subscheme U of X over S , which restricts to the canonical reduction on each fiber of $X \rightarrow S$.*

Now some history. The concept of HN-type was introduced for vector bundles on curves by Harder and Narasimhan in [H-N], where they make use of the corresponding set-theoretic stratification. The existence of a set-theoretic HN-stratification for families of vector bundles (in the sense of μ -semi-stability) over higher dimensional projective varieties was proved by Shatz in [Sh]. Simpson extended these results to pure \mathcal{O} -coherent sheaves (in the sense of Gieseker semistability), and also to pure \mathcal{O} -coherent sheaves of Λ -modules in the sense of Deligne, in [Si]. The concept of canonical reduction for principal bundles was introduced in the context of curves, and its existence and uniqueness proved due to the efforts of Ramanathan [Ram], Atiyah and Bott [A-B], and Behrend [Be 1], [Be 2].

The set-theoretic HN-stratification for pure \mathcal{O} -coherent sheaves on projective schemes, in the sense of Gieseker semistability, was elevated in [Ni 3] to a scheme-theoretic stratification, where the schematic strata have the appropriate universal property, similar to that in Theorem 1.2 above. An analogous result for pure \mathcal{O} -coherent sheaves of Λ -modules on projective schemes was proved in [Gu-Ni 1].

The paper [Gu-Ni 1] also considers families of principal bundles over families of curves in characteristic zero, with reductive structure group, and proves the analog of Theorem 1.2 above over curves. We subsequently learned that this had in fact been earlier proved in the PhD thesis of Behrend [Be 1]. In the present paper, we prove the result in all dimensions. Our proof is by induction on the relative dimension of X/S , which we can now assume is ≥ 2 .

In the case of curves, a deformation theoretic argument shows that the stacks $Bun_{X/S}^\tau(G)$ are all reduced (which holds also in characteristic p if the Behrend conjecture holds for G over k). In higher dimensions, we do not know even in characteristic zero whether $Bun_{X/S}^\tau(G)$ are always reduced.

Remark 1.4 Though we have written this paper for characteristic zero, everything goes through unchanged when k and G are such that the Behrend conjecture holds for G and for all Levi quotients $P/R_u(P)$ of standard parabolics P of G .

Remark 1.5 In the special case $G = GL_{n,k}$, the main results of this paper (that is, Theorems 1.1, 1.2, Corollary 1.3, Proposition 7.4) can be regarded as results about relative HN-filtrations and schematic HN-stratifications for families of vector bundles in the sense of μ -semistability, unlike the corresponding earlier results proved in [Ni 3] which were in the sense of Gieseker semistability. Thus, these results are new even in the case of vector bundles, that is, for $G = GL_{n,k}$.

This article is arranged as follows. The sections 2 to 5 are devoted to setting up the basics of relative canonical reductions and their restrictions to relative divisors $Y \subset X$ over S . The section 6 proves a result on embedding of relative Picard schemes $Pic_{X/S} \rightarrow Pic_{Y/S}$ which is needed for lifting relative canonical reductions from a relative divisor $Y \subset X$ over S to all of X . The main results, including Propositions 7.4, 7.6 and Theorem 1.2, are proved in section 7.

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2 Linearized rational sections of projective bundles

For any vector bundle (that is, a coherent locally free sheaf) F on a scheme X , let $\mathbf{P}(F) = \text{Proj}_{\mathcal{O}_X} \text{Sym}_{\mathcal{O}_X}(F^\vee)$ denote the projective bundle of lines in the fibers of F . Recall that such an F is called a linearization of the projective bundle $\mathbf{P}(F) \rightarrow X$. This motivates the word ‘linearized’ in the following definition.

Definition 2.1 Given a vector bundle F on a scheme X , an F -**linearized rational section** of the projective bundle $\mathbf{P}(F) \rightarrow X$ of lines in the fibers of F will mean a rank 1 locally free subsheaf $[L, f]$ of F , represented by a pair (L, f) consisting of a rank 1 locally free sheaf L on X together with an injective \mathcal{O}_X -linear homomorphism of sheaves $f : L \rightarrow F$, such that the open subscheme $U = \{x \in X \mid \text{rank}(f_x) = 1\}$ is **big in** X , that is, $X - U$ is of codimension ≥ 2 in X at all points (here, $f_x = f|_{\text{Spec } \kappa(x)}$ where $\kappa(x) = \mathcal{O}_{X,x}/\mathfrak{m}_x$ denotes the residue field at x). We denote by $R(F/X)$ the set of all such F -linearized rational sections $[L, f]$.

Let $X \rightarrow S$ be a morphism of schemes, and let F be a vector bundle on X . An F -**linearized relative rational section w.r.t.** S of the projective bundle $\mathbf{P}(F) \rightarrow X$ will mean a rank 1 locally free subsheaf $[L, f]$ of F , represented by a pair (L, f) consisting of a rank 1 locally free sheaf L together with an injective \mathcal{O}_X -linear homomorphism of sheaves $f : L \rightarrow F$, such that the open subscheme $U = \{x \in X \mid \text{rank}(f_x) = 1\}$ is **relatively big in** X **over** S , that is, for each $s \in S$, the closed subset $X_s - U_s$ is of codimension ≥ 2 in X_s at all points of X_s . We will denote by $R(F/X/S)$ the set of all such F -linearized relative rational sections. It is clear that $R(F/X/S) \subset R(F/X)$. When the base S is of the form $\text{Spec } K$ for a field K , then we get $R(F/X/\text{Spec } K) = R(F/X)$.

By definition, two pairs (L_1, f_1) and (L_2, f_2) represent the same element $[L, f]$ in $R(F/X)$ in the absolute case (or in $R(F/X/S)$ in the relative case) if and only if there exists an \mathcal{O}_X -linear isomorphism $\phi : L_1 \rightarrow L_2$ such that $f_1 = f_2 \circ \phi$. Such a ϕ , if it exists, is unique.

Proposition 2.2 *Let $\pi : X \rightarrow S$ be a smooth morphism where S is noetherian, and let F be a vector bundle on X . Let $\mathcal{O}_F(-1) \subset F_{\mathbf{P}(F)}$ denote the tautological line subbundle of the pullback of F to $\mathbf{P}(F)$. Then there is a natural bijection between the following two sets.*

- (i) *The set $R(F/X/S)$ of all F -linearized rational sections of $\mathbf{P}(F)$.*
- (ii) *The set $\Sigma(F/X/S)$ which consists of all pairs (U, σ) where $U \subset X$ is an open subscheme which is relatively big over S and σ is section of $\mathbf{P}(F)$ over U which is maximal in the sense that σ cannot be prolonged to a larger open subscheme, such that the line bundle $\sigma^*(\mathcal{O}_F(-1))$ on U admits a prolongation to a line bundle on X .*

Proof. For any open $U \subset X$, we have a natural bijection between the set of all sections $\sigma : U \rightarrow \mathbf{P}(F)$ and the set of all line subbundles $f' : L' \subset F|_U$, given by pulling back the tautological line subbundle $\mathcal{O}_F(-1) \hookrightarrow F_{\mathbf{P}(F)}$ over $\mathbf{P}(F)$. This defines a natural map $R(F/X/S) \rightarrow \Sigma(F/X/S)$. If L' can be prolonged to a line bundle L on X , then by Lemma 2.3 below, any such a prolongation (that is, a pair $(L, L' \xrightarrow{\sim} L|_U)$) is unique up to a unique isomorphism and moreover if such an L exists then f' admits a unique prolongation to a homomorphism $f : L \rightarrow F$. This shows that the natural map $R(F/X/S) \rightarrow \Sigma(F/X/S)$ is a bijection. \square

Lemma 2.3 *Let $\pi : X \rightarrow S$ be a smooth morphism where S is noetherian, let $j : U \hookrightarrow X$ be an open subscheme which is relatively big over S , and let \mathcal{E} be a locally free \mathcal{O}_X -module. Then the homomorphism $\mathcal{E} \rightarrow j_*(\mathcal{E}|_U)$ is an isomorphism.*

Proof. For any $z \in Z = X - U$, if $\pi(z) = s \in S$, then $\text{depth}(\mathcal{O}_{X_s, z}) \geq 2$ as $\mathcal{O}_{X_s, z}$ is a regular local ring of dimension ≥ 2 . By EGA IV₂ Proposition 6.3.1, we have $\text{depth}(\mathcal{O}_{X, z}) = \text{depth}(\mathcal{O}_{S, s}) + \text{depth}(\mathcal{O}_{X_s, z})$, hence $\text{depth}(\mathcal{O}_{X, z}) \geq 2$. Therefore, $\text{depth}_Z(\mathcal{O}_X) = \inf_{z \in Z} \text{depth}(\mathcal{O}_{X, z}) \geq 2$. Hence the desired conclusion follows from EGA IV₂ Theorem 5.10.5. \square

Remark 2.4 If F is replaced by $F \otimes K$ for a line bundle K on X , then note that we have a natural isomorphism $\mathbf{P}(F) \cong \mathbf{P}(F \otimes K)$ and a compatible natural bijection $R(F/X) \cong R((F \otimes K)/X)$ (or $R(F/X/S) \cong R((F \otimes K)/X/S)$ in the relative case) between the sets of F -linearized and $F \otimes K$ -linearized rational sections of the projective bundles. These natural isomorphisms and natural bijections satisfy the 1-cocycle condition as K varies.

For any vector bundle F on a noetherian integral locally factorial scheme X , we can consider the following three sets.

- (1) The set Σ_η of all **generic sections** $\text{Spec } \kappa(\eta) \rightarrow \mathbf{P}(F)$ of $\pi : \mathbf{P}(F) \rightarrow X$ over the generic point $\eta \in X$.
- (2) The set Σ of all **maximal rational sections**, that is, all pairs (U, σ) where $U \subset X$ is a big open subscheme of X and $\sigma : U \rightarrow \mathbf{P}(F)$ is a section of the

projection $\pi : \mathbf{P}(F) \rightarrow X$ over U , such that (U, σ) is maximal in the sense that σ does not admit a prolongation to a section σ' of π which is defined over a strictly larger open subscheme of X containing U .

(3) The set $R(F/X)$ of all F -linearized rational sections of $\pi : \mathbf{P}(F) \rightarrow X$, defined above.

Proposition 2.5 *For any vector bundle F on a noetherian integral locally factorial scheme X , we have a natural bijection $\Sigma \rightarrow \Sigma_\eta$ which sends (U, σ) to the value of σ at η . Moreover, we have a natural bijection from $R(F/X)$ to Σ which sends (L, f) to the pair (U, σ) , which is well-defined and maximal hence belongs to the set Σ , where $U = \{x \in X \mid \text{rank}(f_x) = 1\}$, and $\sigma : U \rightarrow \mathbf{P}(F)$ is defined by the line subbundle $f|U : L|U \rightarrow F|U$.*

Proof. We will just comment that the assumption of local factoriality of X allows a unique (up to unique isomorphism) extension of a line bundle from a big open subset U to all of X . Moreover, for a line bundle on X , a section over a big U will uniquely extend to a global section because of the same assumption implies normality. The rest of the proof is a routine exercise and we omit the details. \square

2.6 Example of a non-prolongable line bundle. (Due to Najmuddin Fakhruddin.) Let k be any field, let $S = \text{Spec } k[\epsilon]/(\epsilon^2)$, and let $X = \mathbb{P}_S^2$. Let $X_k = \mathbb{P}_k^2$ and let $P_0 \in X_k$ be a closed k -rational point. Let $U = X - \{P_0\}$, which is an open subscheme relatively big over S . The following is an example of a line bundle on U which does not prolong to a line bundle L on X . A Čech calculation using an open cover of $U_k = X_k - \{P_0\}$ by two copies of \mathbb{A}_k^2 shows that $H^1(U_k, \mathcal{O}_{U_k}) \neq 0$. By elementary first order deformation theory (see e.g. [Ni 2] for an expository account), $H^1(U_k, \mathcal{O}_{U_k})$ is the group of isomorphism classes of all line bundles on $U = U_k \times_k \text{Spec } k[\epsilon]/(\epsilon^2)$ whose restriction to U_k is trivial. But as $H^1(X_k, \mathcal{O}_{X_k}) = 0$, any line bundle on $X = X_k \times_k \text{Spec } k[\epsilon]/(\epsilon^2)$ which is trivial on X_k must be trivial. Hence any non-zero element of $H^1(U_k, \mathcal{O}_{U_k})$ defines a line bundle on U which does not prolong to X . More explicitly, $\text{Pic}(U) = \mathbb{Z} \cdot [\mathcal{O}_{X/S}(1)|U] \oplus H^1(U_k, \mathcal{O}_{U_k}) = \text{Pic}(X) \oplus H^1(U_k, \mathcal{O}_{U_k})$, where the second summand is non-zero.

3 Relative rational parabolic reductions

3.1 Some basics about reductive groups

From now onwards, we fix a base field k of characteristic zero, and a connected split reductive group scheme G over k , along with a chosen maximal torus and Borel $T \subset B \subset G$, where T is split over k . We denote the corresponding set of simple roots by $\Delta \subset X^*(T)$, and for each $\alpha \in \Delta$ we denote by $\omega_\alpha \in \mathbb{Q} \otimes X^*(T)$ the

corresponding fundamental dominant weight. Recall that the standard parabolic subgroups P of G (that is, $P \supset B$) are in a one-one correspondence with subsets $I_P \subset \Delta$, where I_P is the set of inverted roots for P .

3.1 Recall that there exist natural internal direct sum decompositions

$$\begin{aligned}\mathbb{Q} \otimes X^*(T) &= (\mathbb{Q} \otimes X^*(P)|_T) \oplus (\oplus_{\alpha \in I_P} \mathbb{Q}\alpha), \text{ where, in turn,} \\ \mathbb{Q} \otimes X^*(P)|_T &= (\oplus_{\alpha \in \Delta - I_P} \mathbb{Q}\omega_\alpha) \oplus (\mathbb{Q} \otimes X^*(G)|_T).\end{aligned}$$

In the above, $X^*(P)|_T$ and $X^*(G)|_T$ denote the images of the homomorphisms $X^*(P) \rightarrow X^*(T)$ and $X^*(G) \rightarrow X^*(T)$ on the character groups which are induced by the inclusions $T \hookrightarrow P$ and $T \hookrightarrow G$.

For simplicity of notation, we will denote the base-change of G (or T or B etc.) to any k -scheme S (in particular, to an extension field K) just by G alone rather than by G_S or G_K , when there is no danger of confusion. Note that for any field extension K/k , we have a natural identification $X^*(T_K) = X^*(T)$ (respectively, $X_*(T_K) = X_*(T)$) on the groups of characters (respectively, on the groups of 1-parameter subgroups). Moreover, note that the set of all parabolics $P \supset B$ in G is in a natural bijection with the set of all parabolics $P' \supset B_K$ under $P \mapsto P_K$, and both these sets are in a natural bijection with the set of all subsets of Δ . Hence for notational simplicity, we will identify any standard parabolic in G_K with the corresponding standard parabolic in G .

Let P be a standard parabolic. Then G/P is projective, and the anti-canonical line bundle $\omega_{G/P}^{-1}$ of G/P is a very ample line bundle which has a natural G -action which lifts the G -action on G/P . The canonical line bundle $\omega_{G/P}$ is the associated line bundle for the principal P -bundle $G \rightarrow G/P$ for a multiplicative character which we denote by $\lambda_P : P \rightarrow \mathbb{G}_m$. The corresponding weight $\lambda_P \in X^*(T)$, which lies in the negative ample cone of G/P , has the form

$$\lambda_P = c \cdot \sum_{\alpha \in \Delta - I_P} \omega_\alpha$$

for some $c > 0$. Then $V_P = \Gamma(G/P, \omega_{G/P}^{-1})^\vee$ is an irreducible G -representation with highest weight λ_P . For any non-zero weight vector $v \in V_P$ with weight λ_P (such a v is unique up to scalar multiple), the isotropy subgroup scheme $I_{[v]} \subset G$ at the point $[v] \in \mathbf{P}(V_P)$ for the G -action on $\mathbf{P}(V)$ is P . This gives a G -equivariant closed imbedding $G/P \hookrightarrow \mathbf{P}(V_P)$, under which $eP \mapsto [v]$. This is just the projective embedding given by the very ample line bundle $\omega_{G/P}^{-1}$.

More generally, for any $\lambda : P \rightarrow \mathbb{G}_m$ which as an element of $X^*(T)$ is a strictly positive linear combination of the ω_α for $\alpha \in \Delta - I_P$, (that is, λ is in the negative ample cone of G/P), we similarly get a G -equivariant closed imbedding $G/P \hookrightarrow \mathbf{P}(V_\lambda)$ where V_λ is an irreducible G -representation with highest weight λ .

Recall that the **standard partial order** on $\mathbb{Q} \otimes X_*(T)$ is defined as follows. If $\mu, \nu \in \mathbb{Q} \otimes X_*(T)$, then we say that $\mu \leq \nu$ if $\langle \chi, \mu \rangle = \langle \chi, \nu \rangle$ for all $\chi \in X^*(G)$

and $\langle \omega_\alpha, \mu \rangle \leq \langle \omega_\alpha, \nu \rangle$ for all simple roots $\alpha \in \Delta$, where $\omega_\alpha \in \mathbb{Q} \otimes X^*(T)$ denotes the fundamental dominant weight corresponding to α , and $\langle \cdot, \cdot \rangle$ denotes the duality pairing. The closed positive Weyl chamber $\overline{C} \subset \mathbb{Q} \otimes X_*(T)$ is the subset consisting of all $\mu \in \mathbb{Q} \otimes X_*(T)$ such that $\langle \alpha, \mu \rangle \geq 0$ for all $\alpha \in \Delta$. The above partial order on $\mathbb{Q} \otimes X_*(T)$ induces the **standard partial order** on \overline{C} .

Remark 3.2 If $I \subset \Delta$ is any subset, then we have the following elementary facts.

(i) Any $\alpha \in \Delta - I$ is expressible as $\alpha = \sum_{\beta \in I} b_\beta \beta + \sum_{\gamma \in \Delta - I} c_\gamma \omega_\gamma$ where each coefficient b_β is ≤ 0 , and

(ii) any fundamental weight ω_α is expressible as $\omega_\alpha = \sum_{\beta \in I} b_\beta \beta + \sum_{\gamma \in \Delta - I} c_\gamma \omega_\gamma$ where all coefficients b_β and c_γ are ≥ 0 .

The statement (i) is the statement labelled (R) in [Gu-Ni 1]. The statement (ii) is immediate from Lemma 1.1 of Raghunathan [Rag], and will be used below.

Lemma 3.3 *Let P be a standard parabolic in G . Let $\mu, \nu \in \overline{C}$, such that $\langle \chi, \mu \rangle = \langle \chi, \nu \rangle = 0$ for all $\chi \in I_P$, and $\langle \chi, \mu \rangle = \langle \chi, \nu \rangle$ for all $\chi \in \widehat{G}$. Then $\mu \leq \nu$ if and only if for all dominant weights $\lambda \in X^*(T)$ which lie in the negative ample cone of G/P , we have $\langle \lambda, \mu \rangle \leq \langle \lambda, \nu \rangle$.*

Proof. By definition, for any $\mu, \nu \in \mathbb{Q} \otimes X_*(T)$, we have $\mu \leq \nu$ if $\langle \chi, \mu \rangle = \langle \chi, \nu \rangle$ for all $\chi \in X^*(G)$ and $\langle \omega_\alpha, \mu \rangle \leq \langle \omega_\alpha, \nu \rangle$ for all simple roots $\alpha \in \Delta$, where $\omega_\alpha \in \mathbb{Q} \otimes X^*(T)$ denotes the fundamental dominant weight corresponding to α . By Remark 3.2, for any $\alpha \in I_P$, the corresponding fundamental dominant weight ω_α can be written as a linear combination $\omega_\alpha = \sum_{\beta \in I_P} b_\beta \beta + \sum_{\gamma \in \Delta - I_P} c_\gamma \omega_\gamma$ where each $c_\gamma \in \mathbb{Q}^{\geq 0}$. As $\sum_{\beta \in I_P} b_\beta \beta$ evaluates to 0 on both μ and ν , we see that $\langle \omega_\alpha, \mu \rangle = \sum_{\gamma \in \Delta - I_P} c_\gamma \langle \omega_\gamma, \mu \rangle$ and $\langle \omega_\alpha, \nu \rangle = \sum_{\gamma \in \Delta - I_P} c_\gamma \langle \omega_\gamma, \nu \rangle$, where each $c_\gamma \geq 0$. Hence to prove that $\mu \leq \nu$ it is necessary and sufficient to prove the following:

(*) If $\alpha \in \Delta - I_P$ then $\langle \omega_\alpha, \mu \rangle \leq \langle \omega_\alpha, \nu \rangle$.

For each $\alpha \in \Delta - I_P$ and each integer $N \geq 0$, we form an element

$$\lambda_{\alpha, N} = N n_0 \omega_\alpha + n_0 \sum_{\beta \in \Delta - I_P} \omega_\beta$$

where $n_0 \in \mathbb{Z}^{>0}$ is so chosen that $n_0 \omega_\alpha \in X^*(T)$ for all $\alpha \in \Delta$. By taking limit as $N \rightarrow \infty$, it follows that in order to prove (*), it is enough to prove the following:

(**) If $\alpha \in \Delta - I_P$ and $N \geq 0$, then $\langle \lambda_{\alpha, N}, \mu \rangle \leq \langle \lambda_{\alpha, N}, \nu \rangle$.

Note that each such element $\lambda_{\alpha, N}$ is a dominant weight in $X^*(T)$, which lies in the negative ample cone of G/P . Hence by hypothesis, the inequality (**) holds. \square

3.2 Principal bundles and linearized rational P -reductions

If $E \rightarrow X$ is a principal G -bundle on a noetherian k -scheme X , then for any standard parabolic P , the G -equivariant closed imbedding $G/P \hookrightarrow \mathbf{P}(V_P)$ defined above gives

rise to a closed embedding

$$i : E/P \hookrightarrow \mathbf{P}(E(V_P))$$

where $E(V_P)$ is the vector bundle on X associated to E by the G -representation V_P , and $\mathbf{P}(E(V_P))$ is the projective bundle of lines in $E(V_P)$. Alternately, we can regard $\mathbf{P}(E(V_P))$ as the associated bundle of E for the G -action on the projective space $\mathbf{P}(V_P)$. More generally, for any dominant weight λ is in the negative ample cone of G/P , we get a closed embedding $E/P \hookrightarrow \mathbf{P}(E(V_\lambda))$ where V_λ is an irreducible G -representation with highest weight λ .

To the above data, we now associate a subset

$$R(E/X, P) \subset R(E(V_P)/X)$$

which consists of all linearized rational sections $[L, f : L \rightarrow E(V_P)]$ of $\mathbf{P}(E(V_P))$ over X such that the section σ of $\mathbf{P}(E(V_P))$ defined by f over the big open subscheme $U = \{x \in X \mid \text{rank}(f_x) = 1\} \subset X$ factors via the closed embedding $E/P \hookrightarrow \mathbf{P}(E(V_P))$ defined above. We call any element $[L, f] \in R(E/X, P)$ as a **linearized rational P -reduction** of the structure group of E . More generally, for any dominant weight λ is in the negative ample cone of G/P , we define a subset $R(E/X, P, \lambda) \subset R(E(V_\lambda)/X)$ by replacing λ_P by λ in the above. We call its elements $(L, f : L \rightarrow E(V_\lambda))$ as **λ -linearized rational P -reductions** of E .

The following proposition is an immediate consequence of Proposition 2.5.

Proposition 3.4 *Let X be a noetherian locally factorial integral scheme over k , let E be a principal G -bundle on X , and P be a standard parabolic in G . Then the restrictions of the natural bijections of Proposition 2.5 gives natural bijections $\Sigma(E/X, P) \rightarrow \Sigma_\eta(E/X, P)$ and $R(E/X, P) \rightarrow \Sigma(E/X, P)$ between the following three sets.*

- (1) *The set $\Sigma_\eta(E/X, P)$ of all **generic P -reductions**, that is, all sections $\text{Spec } \kappa(\eta) \rightarrow E/P$ of $\pi : E/P \rightarrow X$ over the generic point $\eta \in X$.*
- (2) *The set $\Sigma(E/X, P)$ of all **rational P -reductions**, that is, all pairs (P, σ) where $\sigma : U \rightarrow E/P$ is a section of the projection $\pi : E/P \rightarrow X$ over a big open subscheme $U \subset X$, such that (P, σ) is maximal in the sense that σ does not admit a prolongation to a section σ' of π which is defined over a strictly larger open subscheme of X containing U .*
- (3) *The set $R(E/X, P)$ of all linearized rational P -reductions defined above.*

More generally, for any dominant weight λ in the negative ample cone of G/P , restriction gives a bijection $R(E/X, P, \lambda) \rightarrow \Sigma(E/X, P)$. \square

We now come to the relative case, where we make the following definition which is inspired by the Proposition 3.4.

Definition 3.5 Let $X \rightarrow S$ be a morphism of noetherian k -schemes, and let E be a principal G -bundle on X . A **relative linearized rational P -reduction of E w.r.t. S** is a rational linearized P -reduction $[L, f]$ of E over X such that the open subscheme $U = \{x \in X \mid \text{rank}(f_x) = 1\} \subset X$ is relatively big over S . We denote by

$$R(E/X/S, P)$$

the set of all such $[L, f]$. In terms of the Definition 2.1 above, we have

$$R(E/X/S, P) = R(E/X, P) \cap R(E(V_P)/X/S) \subset R(E(V_P)/X).$$

4 Canonical reductions and their restrictions

Let $(X, \mathcal{O}_X(1))$ be a connected smooth projective variety over a field K , together with a very ample line bundle. Let $U \subset X$ be a big open subscheme, and let F be a principal H -bundle on U , where H is a reductive group scheme over K . Recall that F is said to be **semi-stable** if for any parabolic $Q \subset H$, any section $\sigma : W \rightarrow F/Q$ defined on a big open subscheme W of U , and any dominant character $\chi : Q \rightarrow \mathbb{G}_{m,K}$, we have

$$\deg(\chi_* \sigma^* F) \leq 0$$

where $\sigma^* E$ is the principal P -bundle on W defined by the reduction σ , and $\chi_* \sigma^* E$ is the \mathbb{G}_m -bundle obtained by extending its structure group via $\chi : Q \rightarrow \mathbb{G}_m$, which is equivalent to a line bundle on W . This line bundle extends uniquely (up to a unique isomorphism) to a line bundle on X , denoted again by $\chi_* \sigma^* E$, and $\deg(\chi_* \sigma^* F)$ is its degree w.r.t. $\mathcal{O}_X(1)$.

Now let K be an extension field of k , let $(X, \mathcal{O}_X(1))$ be a connected smooth projective variety over K , and let E be a principal G -bundle on X . Recall that a **canonical reduction** of E is a rational P -reduction (P, σ) of structure group of E to a standard parabolic $P \subset G$ (see Proposition 3.4.(2)) for which the following two conditions (C-1) and (C-2) hold:

(C-1) If $\rho : P \rightarrow L = P/R_u(P)$ is the Levi quotient of P (where $R_u(P)$ is the unipotent radical of P) then the principal L -bundle $\rho_* \sigma^* E$ is a semistable principal L -bundle defined on the big open subscheme U on which σ is defined.

(C-2) For any non-trivial character $\chi : P \rightarrow \mathbb{G}_m$ whose restriction to the chosen maximal torus $T \subset B \subset P$ is a non-negative linear combination $\sum n_i \alpha_i$ of simple roots $\alpha_i \in \Delta$ (where $n_i \geq 0$, and at least one $n_i \neq 0$), we have $\deg(\chi_* \sigma^* E) > 0$.

Recall that the **type** of any rational P -reduction $\sigma : U \rightarrow E/P$ of E is the element $\mu_{(P,\sigma)}(E) \in \mathbb{Q} \otimes X_*(T)$ defined w.r.t. decomposition given in statement 3.1 by

$$\langle \chi, \mu_{(P,\sigma)}(E) \rangle = \begin{cases} \deg(\chi_* \sigma^* E) & \text{if } \chi \in X^*(P)|_T, \\ 0 & \text{if } \chi \in I_P. \end{cases}$$

In particular, if (P, σ) is a canonical reduction, then the **Harder-Narasimhan type** $\text{HN}(E)$ is defined to be $\mu_{(P, \sigma)}(E) \in \mathbb{Q} \otimes X_*(T)$. This is well-defined, as a canonical reduction is unique. It can be shown that $\text{HN}(E) \in \overline{\mathcal{C}} \subset \mathbb{Q} \otimes X_*(T)$.

4.1 Maximality property of the canonical reduction. Let E be a principal G -bundle on a smooth connected projective variety $(X, \mathcal{O}_X(1))$ and let $\sigma : U \rightarrow E/P$ define a rational reduction to a standard parabolic P . Then $\mu_{(P, \sigma)} \leq \text{HN}(E)$ in $\mathbb{Q} \otimes X_*(T)$, and if equality holds then $\sigma : U \rightarrow E/P$ defines the canonical reduction.

Lemma 4.2 (Fundamental Inequality.) *Let $(X, \mathcal{O}_X(1))$ be a smooth connected projective variety over an extension field K of k . Let E be a principal G -bundle on X , let P be a standard parabolic in G , and let $\sigma : U \rightarrow E/P$ be a section over a big open subscheme $U \subset X$. Suppose that U is the maximal open subscheme to which σ can prolong. Let $\mu_{(P, \sigma)}(E) \in \mathbb{Q} \otimes X_*(T)$ be the type of the reduction $\sigma : U \rightarrow E/P$. Let $Y \subset X$ be any smooth connected hypersurface such that $Y \in |\mathcal{O}_X(m)|$ for some $m \geq 1$ (note that $m = \deg(Y)/\deg(X)$). Let $U' \subset Y$ be a big open subscheme of Y such that $U \cap Y \subset U'$, and let $\sigma' : U' \rightarrow (E|Y)/P$ be a section which prolongs $\sigma|_{(U \cap Y)}$ (such a prolongation of σ exists, as follows by applying the valuative criterion for properness to $(E|Y)/P \rightarrow Y$). Let $\mu_{(P, \sigma')}(E|Y) \in \mathbb{Q} \otimes X_*(T)$ be the type of the reduction $\sigma' : U' \rightarrow (E|Y)/P$. Then we have*

$$m \cdot \mu_{(P, \sigma)}(E) \leq \mu_{(P, \sigma')}(E|Y).$$

Moreover, in the above we have equality $m \cdot \mu_{(P, \sigma)}(E) = \mu_{(P, \sigma')}(E|Y)$ if and only if $U \cap Y$ is big in Y .

Proof. By definition of the type of a rational parabolic reduction, we have $\langle \chi, m \cdot \mu_{(P, \sigma)}(E) \rangle = \langle \chi, \mu_{(P, \sigma')}(E|Y) \rangle = 0$ for all $\chi \in I_P$.

If $\chi \in \widehat{G}$, then for any rational parabolic reduction (P, σ) of E and (P, σ') of $E|Y$, we have isomorphisms of line bundles $\chi_*(E) \cong \chi_*(\sigma^*(E))$ over X , and $\chi_*(E)|Y \cong \chi_*(E|Y) \cong \chi_*(\sigma'^*(E|Y))$ over Y . Also, for any line bundle L on X , we have $\deg_Y(L|Y) = m \cdot \deg_X(L)$. Hence

$$\langle \chi, m \cdot \mu_{(P, \sigma)}(E) \rangle = m \cdot \deg_X(\chi_*(E)) = \deg_Y(\chi_*(E|Y)) = \langle \chi, \mu_{(P, \sigma')}(E|Y) \rangle.$$

Hence by Lemma 3.3, the inequality $m \cdot \mu_{(P, \sigma)}(E) \leq \mu_{(P, \sigma')}(E|Y)$ will follow if we prove the following:

(***) If $\lambda \in X^*(T)$ is a dominant weight which lies in the negative ample cone of G/P , then $m \cdot \langle \lambda, \mu_{(P, \sigma)}(E) \rangle \leq \langle \lambda, \mu_{(P, \sigma')}(E|Y) \rangle$.

To see this, let L denote the line bundle $\lambda_*\sigma^*(E)$ on X . Under the bijection given by Proposition 3.4, the rational P -reduction (P, σ) corresponds to $[L, f] \in R(E/X, P, \lambda)$ where $f : L = \lambda_*\sigma^*(E) \rightarrow E(V_\lambda)$ is an injective \mathcal{O}_X -linear homomorphism, and

$$U = \{x \in X \mid \text{rank}(f_x) = 1\}.$$

Let $Y \subset X$ be any smooth connected hypersurface such that $Y \in |\mathcal{O}_X(m)|$ for some $m \geq 1$ (note that $m = \deg(Y)/\deg(X)$). Let $U' \subset Y$ be a big open subscheme of Y such that $U \cap Y \subset U'$, and let $\sigma' : U' \rightarrow (E|Y)/P$ be a section with

$$\sigma'|_{(U \cap Y)} = \sigma|_{(U \cap Y)}.$$

Such a pair (U', σ') indeed exists, as follows by applying the valuative criterion for properness to $(E|Y)/P \rightarrow Y$. Moreover, we can take $U' \subset Y$ to be the maximal open subscheme to which $\sigma' : U' \rightarrow (E|Y)/P$ prolongs.

By Proposition 3.4, the rational P -reduction (P, σ') of $E|Y$ corresponds to $[L', f'] \in R((E|Y)/Y, P, \lambda)$ where $f' : L' = \lambda_* \sigma'^*(E|Y) \rightarrow (E|Y)(V_\lambda)$ is an injective \mathcal{O}_Y -linear homomorphism, and

$$U' = \{y \in Y \mid \text{rank}(f'_y) = 1\}.$$

As σ' prolongs $\sigma|_{(U \cap Y)}$, it follows that the homomorphism $f|Y : L|Y \hookrightarrow (E|Y)(V_\lambda)$ factors uniquely via $f' : L' \hookrightarrow (E|Y)(V_\lambda)$ to give an injective \mathcal{O}_Y -linear homomorphism

$$\varphi : L|Y \hookrightarrow L'$$

which is fiberwise injective on $U \cap Y$. It now follows from Remark 4.3 below that

$$\deg(L|Y) \leq \deg(L')$$

and equality holds if and only if the set $Z = \{y \in Y \mid \varphi_y = 0\}$ is of codimension ≥ 2 in Y . As $f_y = f'_y \circ \varphi_y$, we have $Z = Y - (U \cap Y)$. Hence the equality $\deg(L|Y) = \deg(L')$ holds if and only if $U \cap Y$ is big in Y .

Now note that $\deg(L|Y) = m \cdot \deg(L)$, and $\deg(L) = \langle \lambda, \mu_{(P, \sigma)}(E) \rangle$ while $\deg(L') = \langle \lambda, \mu_{(P, \sigma')}(E|Y) \rangle$. Hence we get $m \cdot \langle \lambda, \mu_{(P, \sigma)}(E) \rangle \leq \langle \lambda, \mu_{(P, \sigma')}(E|Y) \rangle$. This completes the proof of (***) .

Moreover, as $\deg(L|Y) = \deg(L')$ if and only if $U \cap Y$ is big in Y , we see that $m \cdot \langle \lambda, \mu_{(P, \sigma)}(E) \rangle = \langle \lambda, \mu_{(P, \sigma')}(E|Y) \rangle$ if and only if $U \cap Y$ is big in Y . Hence, Lemma 4.2 is proved. \square

Remark 4.3 Degree inequality for line bundles. Let $(Y, \mathcal{O}_Y(1))$ be an irreducible smooth projective variety over a field K , together with a very ample line bundle. If $L_1 \subset L_2$ are coherent \mathcal{O}_Y -modules such that both L_1 and L_2 are invertible sheaves (line bundles), then

$$\deg(L_1) \leq \deg(L_2).$$

Moreover, equality holds in the above if and only if $L_1 = L_2$. We leave the proof of this elementary fact to the reader.

Proposition 4.4 HN-type rises under restriction. *Let $(X, \mathcal{O}_X(1))$ be a smooth connected projective variety over an extension field K of k , and let $Y \subset X$ be any*

smooth connected hypersurface such that $Y \in |\mathcal{O}_X(m)|$ for some $m \geq 1$. Let E be a principal G -bundle on X . Then we have

$$m \cdot \text{HN}(E) \leq \text{HN}(E|Y).$$

Moreover, in the above we have equality

$$m \cdot \text{HN}(E) = \text{HN}(E|Y)$$

only if for the canonical reduction $\sigma : U \rightarrow E/P$ of E , the intersection $U \cap Y$ is big in Y , and $\sigma|(U \cap Y) : U \cap Y \rightarrow (E|Y)/P$ represents the canonical reduction of $E|Y$.

Proof. Let the section $\sigma : U \rightarrow E/P$ define a canonical reduction for E , where $U \subset X$ is a big open subscheme. Moreover, assume that U is the maximal open subscheme of X to which σ prolongs. Then using the notation of Lemma 4.2, we have

$$\begin{aligned} m \cdot \text{HN}(E) &= m \cdot \mu_{(P,\sigma)}(E) \text{ by definition of HN-type,} \\ &\leq \mu_{(P,\sigma')}(E|Y) \text{ by Lemma 4.2,} \\ &\leq \text{HN}(E|Y) \text{ by the maximality property 4.1 of the HN-type.} \end{aligned}$$

By the last part of Lemma 4.2, equality holds in the above if and only if $U \cap Y$ is big in Y . Now by the maximality property of the canonical reduction (see statement 4.1), it follows that $\sigma|(U \cap Y) : U \cap Y \rightarrow (E|Y)/P$ represents the canonical reduction of $E|Y$. This completes the proof of Proposition 4.4. \square

By combining the analog for principal bundles of the semistable restriction theorem of Mehta and Ramanathan with the Proposition 4.4 above, we get the following result for HN-types of restrictions of principal bundles.

Proposition 4.5 (Mehta-Ramanathan restriction theorem for HN-types.)

Let $(X, \mathcal{O}_X(1))$ be a smooth connected projective variety over an extension field K of k , of dimension ≥ 2 . Let E be a principal G -bundle on X . Then there exists an integer $m_0 \geq 1$ such that for any general smooth hypersurface $Y \subset X$ where $Y \in |\mathcal{O}_X(m)|$ where $m \geq m_0$, the HN-types $\text{HN}(E)$ and $\text{HN}(E|Y)$ are related by

$$m \cdot \text{HN}(E) = \text{HN}(E|Y).$$

Moreover, if the canonical reduction of E is the rational reduction (U, σ) to a standard parabolic P , then $U \cap Y$ is big in Y , and the canonical reduction of $E|Y$ is represented by the rational section of $(E|Y)/P$ defined by $\sigma|(U \cap Y)$.

5 Restriction of relative canonical reduction

Let $X \rightarrow S$ be a smooth projective morphism of noetherian k -schemes with geometrically connected fibers, with a given relatively very ample line bundle $\mathcal{O}_{X/S}(1)$ on X .

Definition 5.1 Let E be a principal G -bundle on X , and suppose that the HN-type is constant over S , that is, there is some $\tau \in \overline{C}$ such that $\text{HN}(E_s) = \tau$ for all $s \in S$. A **relative canonical reduction of E w.r.t. S** will mean a relative linearized rational P -reduction $f : L \rightarrow E(V_P)$ of E w.r.t. S such that for each $s \in S$, the restriction $f_s : L_s \rightarrow E_s(V_P)$ is a canonical reduction of E_s on $(X_s, \mathcal{O}_{X_s}(1))$ over the base field $\kappa(s)$.

Proposition 5.2 *With $X \rightarrow S$ as above, let E be a principal G -bundle on X with constant HN-type $\text{HN}(E_s) = \tau$ for all $s \in S$. Suppose that we are given a relative canonical reduction (L, f) of E w.r.t. S , as defined above. Let $Y \subset X$ be a relative effective divisor such that $\mathcal{O}_X(Y) \cong \mathcal{O}_{X/S}(m)$, and such that Y is smooth over S . Suppose that $\text{HN}(E_s|Y_s) = m \cdot \tau$ for all $s \in S$. Then $(L|Y, f|Y)$, where $f|Y : L|Y \rightarrow E(V_P)|Y = (E|Y)(V_P)$ denotes the restriction of f to Y , is a relative canonical reduction of the principal G -bundle $E|Y$ on Y w.r.t. S .*

Proof. Let $U \subset X$ be the subset where $\text{rank}(f_x) = 1$. Let $\sigma : U \rightarrow E/P$ be the section defined by f . For any $s \in S$, we get a big open subset $U_s = U \cap X_s$ and a section $\sigma_s = \sigma|_{U_s} : U_s \rightarrow E_s/P$. Note that σ_s is defined by $f_s : L_s \rightarrow E_s(V_P)$. By Proposition 3.4, (U_s, σ_s) is maximal. By assumption, $\text{HN}(E_s|Y_s) = m \cdot \text{HN}(E)$. Hence by Proposition 4.4, $U_s \cap Y_s$ is big in Y_s . Hence $f|Y : L|Y \rightarrow (E|Y)(V_P)$ is a relative canonical reduction of $E|Y$. \square

Remark 5.3 In the light of Proposition 5.2, if Theorem 1.2 is true, then $S^\tau(E)$ must be a closed subscheme of $S^{m\tau}(E|Y)$. The scheme $S^{m\tau}(E|Y)$ exists by inductive hypothesis on relative dimension. In what follows, we show that there indeed exists a closed subscheme of $S^{m\tau}(E|Y)$ which has the universal property required of $S^\tau(E)$.

6 Embeddings of relative Picard schemes

We will use repeatedly the following result of Mumford.

Proposition 6.1 (Use of m -regularity.) *Let S be any noetherian scheme, and let $\pi : X \rightarrow S$ be a projective morphism, together with a relatively very ample line bundle $\mathcal{O}_{X/S}(1)$. Let \mathcal{F} be any coherent sheaf on X . Then there exists an integer m such that for all $s \in S$, the sheaf $\mathcal{F}_s = \mathcal{F}|_{X_s}$ is m -regular, that is, $H^i(X_s, \mathcal{F}_s(m-i)) = 0$ for all $i \geq 1$.*

Proof (sketch). (See Mumford [Mu], or [Ni 1] for an expository account.) There exists a surjection $\mathcal{O}_X^n(-N) \rightarrow \mathcal{F}$, for some integers n and N . Only finitely many different Hilbert polynomials occur as Hilbert polynomials of \mathcal{F}_s as s varies over S . Hence now the conclusion follows by Mumford's theorem (see page 101 of [Mu] or Theorem 5.3 of [Ni 1]). \square

Proposition 6.2 *Let $X \rightarrow S$ be a smooth projective morphism of schemes of relative dimension ≥ 2 , with geometrically connected fibers, where S is noetherian, and let $\mathcal{O}_{X/S}(1)$ be a relatively very ample line bundle on X/S . Then there exists $m_0 \in \mathbb{Z}$ with the following property: If $Y \subset X$ is a relative effective divisor which is smooth over S and such that $\mathcal{O}_X(Y) \simeq \mathcal{O}_{X/S}(m)$ where $m \geq m_0$, then the morphism*

$$r : \text{Pic}_{X/S} \rightarrow \text{Pic}_{Y/S}$$

of relative Picard schemes which is induced by the inclusion $Y \hookrightarrow X$ is a closed embedding.

Proof. We will assume the basics of Grothendieck's theory of relative Picard schemes, as explained in Kleiman [K]. For any integer d , let $\text{Pic}_{X/S}^d \subset \text{Pic}_{X/S}$ and $\text{Pic}_{Y/S}^d \subset \text{Pic}_{Y/S}$ denote the open and closed ('clopen') subschemes corresponding to line bundles of degree d w.r.t. the given relatively very ample line bundle $\mathcal{O}_{X/S}(1)$ on X/S and its restriction to Y/S . Note that $\text{Pic}_{X/S}$ and $\text{Pic}_{Y/S}$ are disjoint unions of the subschemes $\text{Pic}_{X/S}^d$ and $\text{Pic}_{Y/S}^d$ respectively. The S -morphism $r : \text{Pic}_{X/S} \rightarrow \text{Pic}_{Y/S}$ maps $\text{Pic}_{X/S}^d$ into $\text{Pic}_{Y/S}^{md}$. Each restriction $r^d : \text{Pic}_{X/S}^d \rightarrow \text{Pic}_{Y/S}^{md}$ of r is proper, as both $\text{Pic}_{X/S}^d$ and $\text{Pic}_{Y/S}^{md}$ are projective over S by the Kleiman boundedness theorem. Hence it follows that $r : \text{Pic}_{X/S} \rightarrow \text{Pic}_{Y/S}$ is proper. Moreover, if we show that each r^d is a closed embedding, it would follow that so is r .

It is enough to prove the proposition after base change under an fppf (or surjective étale) morphism to S , as relative Picard schemes base change correctly under any morphism, and a morphism is a closed embedding if and only if its pullback under an fppf base change is a closed embedding. Note that $Y \rightarrow S$ is an fppf morphism, and moreover, the base-change $Y \times_S Y \rightarrow Y$ of $Y \rightarrow S$ under $Y \rightarrow S$ admits a global section, namely, the diagonal. Hence without loss of generality, we can assume that $Y \rightarrow S$ admits a global section. Using this global section for normalization, we see that there exists a Poincaré line bundle \mathbb{L} on $X \times_S \text{Pic}_{X/S}^0$.

Let $\omega_{X/S} = \Omega_{X/S}^n$ be the rank 1 locally free sheaf on X of top relative exterior forms, where n denotes the fiber dimension of X/S . Let $p_1 : X \times_S \text{Pic}_{X/S}^0 \rightarrow X$ and $p_2 : X \times_S \text{Pic}_{X/S}^0 \rightarrow \text{Pic}_{X/S}^0$ denote the projections. Consider the line bundles $\mathbb{L} \otimes p_1^* \omega_{X/S}$ and $\mathbb{L}^{-1} \otimes p_1^* \omega_{X/S}$ on $X \times_S \text{Pic}_{X/S}^0$. As $\text{Pic}_{X/S}^0$ is noetherian and as $p_2 : X \times_S \text{Pic}_{X/S}^0 \rightarrow \text{Pic}_{X/S}^0$ is projective, by Mumford's theorem on m -regularity (Proposition 6.1) applied to $\mathbb{L} \otimes p_1^* \omega_{X/S}$ and $\mathbb{L}^{-1} \otimes p_1^* \omega_{X/S}$, there exists an integer m_1 such that for all $m \geq m_1$, for any $z \in \text{Pic}_{X/S}^0$ and for all $i \geq 1$ we have

$$H^i(X_z, \mathbb{L}_z^{-1} \otimes \omega_{X_z/z}(m)) = H^i(X_z, \mathbb{L}_z \otimes \omega_{X_z/z}(m)) = 0,$$

where $X_z = X \times_S z \subset X \times_S \text{Pic}_{X/S}^0$ and $\mathbb{L}_z = \mathbb{L}|_{X_z}$. Hence by Grothendieck duality on X_z we have

$$H^{n-i}(X_z, \mathbb{L}_z(-m)) = H^{n-i}(X_z, \mathbb{L}_z^{-1}(-m)) = 0$$

for all $z \in \text{Pic}_{X/S}^0$, $m \geq m_1$ and $i \geq 1$. Next by a similar argument for $X \rightarrow S$, we see that there exists an integer m_2 such that

$$H^{n-i}(X_s, \mathcal{O}_{X_s}(-m)) = 0 \text{ for all } s \in S, m \geq m_2 \text{ and } i \geq 1.$$

We will show that the integer $m_0 = \max\{m_1, m_2\}$ has the property required by the proposition.

By Lemma 6.3, in order to prove the proposition it is enough to prove that given any algebraically closed field K with a morphism $\text{Spec } K \rightarrow S$, the base change $r_K^d : \text{Pic}_{X_K/K}^d \rightarrow \text{Pic}_{Y_K/K}^{md}$ is a closed embedding. As this morphism is a translate of r_K^0 by a K -point of $\text{Pic}_{X_K/K}^d$, it is enough to show that the proper morphism $r_K^0 : \text{Pic}_{X_K/K}^0 \rightarrow \text{Pic}_{Y_K/K}^0$ is a closed embedding.

We first show that the abstract group homomorphism $r_K^0(K) : \text{Pic}_{X_K/K}^0(K) \rightarrow \text{Pic}_{Y_K/K}^0(K)$ is injective. As K is algebraically closed, this is just the restriction map $\text{Pic}^0(X_K) \rightarrow \text{Pic}^0(Y_K)$. If a point $P \in \text{Pic}_{X_K/K}^0(K)$ is the isomorphism class of a line bundle L on X_K , then note that L is isomorphic to the base change of the Poincaré bundle \mathbb{L} on $X \times_S \text{Pic}_{X/S}$ under the morphism $P : \text{Spec } K \rightarrow \text{Pic}_{X/S}$. Note that Y_K is smooth projective over K , and in fact Y_K is connected as the relative dimension of X/S is ≥ 2 (see for example [H-AG] 3.7.9). Hence if $L|_{Y_K}$ is trivial, then

$$\dim_K H^0(Y_K, L|_{Y_K}) = \dim_K H^0(Y_K, L^{-1}|_{Y_K}) = 1.$$

But from our choice of m_0 , for each $m \geq m_0$ we have the cohomology vanishings $H^0(X_K, L(-m)) = H^1(X_K, L(-m)) = 0$ and $H^0(X_K, L^{-1}(-m)) = H^1(X_K, L^{-1}(-m)) = 0$. Hence from the long exact cohomology sequences for the short exact sequences $0 \rightarrow L(-m) \rightarrow L \rightarrow L|_{Y_K} \rightarrow 0$ and $0 \rightarrow L^{-1}(-m) \rightarrow L^{-1} \rightarrow L^{-1}|_{Y_K} \rightarrow 0$, it follows that the restriction maps $H^0(X_K, L) \rightarrow H^0(Y_K, L|_{Y_K})$ and $H^0(X_K, L^{-1}) \rightarrow H^0(Y_K, L^{-1}|_{Y_K})$ are bijections. This shows that

$$\dim_K H^0(X_K, L) = \dim_K H^0(X_K, L^{-1}) = 1,$$

hence L is trivial, as X_K is integral and proper over K .

Hence $r_K^0 : \text{Pic}_{X_K/K}^0 \rightarrow \text{Pic}_{Y_K/K}^0$ is a proper homomorphism of finite type group schemes over $K = \overline{K}$, which is injective on K -valued points. Hence its schematic kernel N_K is a finite connected scheme, supported over the identity $e = \text{Spec } K \subset \text{Pic}_{X_K/K}^0$. To show that N_K is the identity, it is thus enough to show that the tangent map $d_e r_K^0 : T_e \text{Pic}_{X_K/K}^0 \rightarrow T_e \text{Pic}_{Y_K/K}^0$ is injective. But this is just the restriction map $H^1(X_K, \mathcal{O}_{X_K}) \rightarrow H^1(Y_K, \mathcal{O}_{Y_K})$, which is injective for $m \geq m_0$ as its kernel $H^1(X_K, \mathcal{O}_{X_K}(-m))$ is zero, so $N_K = \text{Spec } K$. This completes the proof that r_K^0 is a closed embedding, and so the Proposition 6.2 is proved. \square

Lemma 6.3 *Let S be a noetherian scheme, and let $r : P \rightarrow Q$ be a proper morphism of S -schemes. If for each $s \in S$, the restriction $r_s : P_s \rightarrow Q_s$ is a closed embedding, then r is a closed embedding. Equivalently, if for each algebraically closed field K with a morphism $\text{Spec } K \rightarrow S$, if the base-change $r_K : P_K \rightarrow Q_K$ is a closed embedding, then r is a closed embedding.*

Proof. As r is proper, $r_*\mathcal{O}_P$ is coherent, and so the cokernel C of $r^\# : \mathcal{O}_Q \rightarrow r_*\mathcal{O}_P$ is a coherent sheaf of \mathcal{O}_Q -modules. Being injective and proper, r is finite hence affine. As r_s is a closed embedding for each $s \in S$, we must have $C_s = 0$ on Q_s for each $s \in S$. Hence by the coherence of C , we get $C = 0$, which shows that $r^\#$ is surjective. This completes the proof of the lemma. \square

Remark 6.4 Let $Y \subset X \rightarrow S$ be as before, so that we have a closed embedding $r : \text{Pic}_{X/S} \hookrightarrow \text{Pic}_{Y/S}$. For any line bundle L_Y on Y , let $S_1 \subset S$ be the closed subscheme which is the schematic inverse image of $\text{Pic}_{X/S} \subset \text{Pic}_{Y/S}$ under the section $[L_Y] : S \rightarrow \text{Pic}_{Y/S}$. It has the universal property that for any morphism $T \rightarrow S$, there exists a line bundle on X_T which lifts (prolongs) the line bundle $\phi^*(L_T)$ from Y_T under the inclusion $Y_T \hookrightarrow X_T$ if and only if $T \rightarrow S$ factors via $S_1 \hookrightarrow S$.

7 Proofs of the main results

We will need the following two facts (Theorem 7.1 and Lemma 7.2) which are known to experts and can be found in more general forms in the literature (self-contained proofs are given in [Gu-Ni 2]).

Theorem 7.1 (Vanishing theorem for $R^i\pi_*F(-m)$ for $i \leq n - 1$.) *Let S be a noetherian scheme, let $\pi : X \rightarrow S$ be smooth projective, and let $\mathcal{O}_{X/S}(1)$ be a relatively ample line bundle on X/S . Suppose that all fibers X_s are connected, of constant dimension n . Then for any locally free sheaf F on X , there exists an $m_0 \in \mathbb{Z}$ such that for all $m \geq m_0$ and for all morphisms $T \rightarrow S$ where T is noetherian, we have $R^i(\pi_T)_*(F_T(-m)) = 0$ for all $i \leq n - 1$, where F_T is the pullback of F to $X_T = X \times_S T$ and $\pi_T : X_T \rightarrow T$ is the projection.* \square

Lemma 7.2 *Let S be a noetherian scheme and let $\pi : X \rightarrow S$ be a proper flat surjective morphism. Suppose moreover that each fiber of π is geometrically integral. Then the homomorphism $\pi^\# : \mathcal{O}_S \rightarrow \pi_*\mathcal{O}_X$ is an isomorphism.* \square

Remark 7.3 Let S be a topological space, and let (\mathcal{P}, \leq) be a partially ordered set. We will say that a map $h : S \rightarrow \mathcal{P}$ is **upper semicontinuous** if for each $\tau \in \mathcal{P}$, the subset $\{s \in S \mid h(s) \leq \tau\}$ is open in S . It can be seen that a map $h : S \rightarrow \mathcal{P}$ is upper semicontinuous in the above sense if and only if for each $s_0 \in S$, there exists

an open neighbourhood $s_0 \subset U \subset S$ such that $h|_U$ takes a maximum at $s_0 \in U$, that is, $h(u) \leq h(s_0)$ for all $u \in U$. Consequently, we have $\overline{S^\tau} \subset \cup_{\tau' \geq \tau} S^{\tau'}$.

We now revert to the notation where k is a given field of characteristic 0, and G is a given split reductive group scheme over k together with a given split maximal torus and a Borel $T \subset B \subset X$.

Proposition 7.4 (Semicontinuity and uniqueness.) *Let $X \rightarrow S$ be a smooth projective morphism of noetherian k -schemes with geometrically connected fibers, together with a given relatively very ample line bundle $\mathcal{O}_{X/S}(1)$ on X/S . Let E be a principal G -bundle on X . Then we have the following.*

- (1) *The function $S \rightarrow \overline{C} : s \mapsto \text{HN}(E_s)$ is upper semicontinuous. In particular, the semistable locus $S^{ss} = \{s \in S \mid \text{HN}(E_s) = 0\}$ is open in S .*
- (2) *If the HN-type $\text{HN}(E_s)$ is constant on S , then there exists at most one relative canonical reduction for E .*

Proof. The result is proved in [Gu-Ni 1] when X/S has relative dimension 1. We now proceed by induction on the relative dimension, which we assume is ≥ 2 . By application of m -regularity (see Proposition 6.1), there exists an integer m_1 that each \mathcal{O}_{X_s} is m_1 -regular for all $s \in S$. Hence by Grothendieck's theorem on semicontinuity and base change (see Theorem 5.10 in [Ni 1] for a formulation directly useful here), for all $m \geq m_1$, the direct image $\pi_*(\mathcal{O}_{X/S}(m))$ is locally free, the higher direct images $R^i\pi_*(\mathcal{O}_{X/S}(m))$ are equal to 0 for all $i \geq 1$, and for all $s \in S$, the evaluation homomorphism $(\pi_*\mathcal{O}_{X/S}(m))_s \rightarrow H^0(X_s, \mathcal{O}_{X_s}(m))$ is surjective.

For any $s \in S$ and a smooth effective divisor $H \subset X_s$ such that $H \in |\mathcal{O}_{X_s}(m)|$, let $v \in H^0(X_s, \mathcal{O}_{X_s}(m))$ define H . As $\pi_*(\mathcal{O}_{X/S}(m))$ is locally free and as $(\pi_*\mathcal{O}_{X/S}(m))_s \rightarrow H^0(X_s, \mathcal{O}_{X_s}(m))$ is surjective, there exists an open neighbourhood U of s in S and a section $u \in (\pi_*\mathcal{O}_{X/S}(m))(U)$ such that $u_s = v$. Let $Y \subset \pi^{-1}(U)$ be the subscheme defined by the vanishing of u . As $v \in H^0(X_s, \mathcal{O}_{X_s}(m))$ is a regular section, it follows by using the local criterion of flatness (see Lemma 9.3.4 of [K]) that $Y \rightarrow U$ is flat at all points of Y_s , hence by properness $Y \rightarrow U$ is flat over a neighbourhood of s . As $Y \rightarrow U$ is smooth over s , there exists a possibly smaller neighbourhood $s \in V \subset U$ such that $Y_V \rightarrow V$ is smooth. Then $Y \rightarrow V$ is a relative effective divisor in $|\mathcal{O}_{X_V/V}(m)|$, with $Y_s = H$. This shows how to prolong the smooth effective divisor $H \subset X_s$ to a smooth relative effective divisor in an open neighbourhood V of s in the linear system defined by $\mathcal{O}_{X_V/V}(m)$.

Proof of 7.4(1): By Remark 7.3, it is enough to prove that for any $s_0 \in S$, the value $\tau = \text{HN}(E_{s_0})$ is a local maximum for $\text{HN}(E_s)$. Given $s_0 \in S$, by the Mehta-Ramanathan theorem (Proposition 4.5), there exists an integer m_0 such that for any $m \geq m_0$, for a general smooth hypersurface $H \subset X_{s_0}$ in the linear system $|\mathcal{O}_{X_0}(m)|$, the restriction $E_{s_0}|_H$ has HN-type $m \cdot \tau$. Let $m_{s_0} = \max\{m_0, m_1\}$ where m_1 was chosen at the beginning of the proof. Let $m \geq m_{s_0}$, and let $H \subset X_{s_0}$ be

a hypersurface in $|\mathcal{O}_{X_0}(m)|$ which is connected and smooth over $\text{Spec } \kappa(s_0)$ (which exists), with $\text{HN}(E_{s_0}|H) = m \cdot \tau$.

As we have shown above, after shrinking S to a smaller open neighbourhood of s_0 if needed, there exists a smooth relative effective divisor $Y \subset X$ over S (where for simplicity, the same notation S now denotes an open neighbourhood of s_0 in the original S), such that $\mathcal{O}_X(Y) \cong \mathcal{O}_{X/S}(m)$, and such that

$$Y_{s_0} = H \subset X_{s_0}.$$

As Y/S is of relative dimension less than that of X/S , by inductive hypothesis on the relative dimension, the result holds for $E|Y$, hence the value $m \cdot \tau$ at s_0 is a local maximum value for the function $s \mapsto \text{HN}(E|Y_s)$. Hence by shrinking S to a further smaller open neighbourhood of s_0 (which we again denote just by S for simplicity), we can assume that $m \cdot \tau$ is the maximum value of $\text{HN}(E|Y_s)$ on S . By Proposition 4.4, for any $s \in S$ we have $m \cdot \text{HN}(E_s) \leq \text{HN}(E|Y_s)$. Hence for all $s \in S$, we get the inequalities

$$m \cdot \text{HN}(E_s) \leq \text{HN}(E|Y_s) \leq \text{HN}(E|H) = m \cdot \text{HN}(E_{s_0})$$

showing that $\text{HN}(E_{s_0})$ is a local maximum for $\text{HN}(E_s)$, as claimed. This completes the proof of Proposition 7.4.(1).

Proof of 7.4.(2): Let P be the unique standard parabolic in G such that τ lies in the negative ample cone of G/P . We must show that if there exists two relative canonical reductions $(L, f : L \rightarrow E(V_P))$ and $(L', f' : L' \rightarrow E(V_P))$ for E , then there exists an isomorphism $\phi : L \rightarrow L'$ such that $f = f' \circ \phi$.

Note that such an isomorphism ϕ , if it exists, is automatically unique as f' is an injective homomorphism of sheaves. Hence, it is enough to prove the existence of ϕ after base change to an fppf cover of S , as the resulting ϕ will satisfy the cocycle condition by uniqueness, hence would descend to S . In particular, it is enough to prove the result in a neighbourhood of each point $s_0 \in S$.

By the proof of part (1) above, after possibly shrinking S to a smaller open neighbourhood of s_0 (which we again denote just by S), for any $m \geq m_{s_0}$ there exists a smooth relative effective divisor $Y \subset X$ over S , such that $\mathcal{O}_X(Y) \cong \mathcal{O}_{X/S}(m)$, and such that the inequalities $m \cdot \text{HN}(E_s) \leq \text{HN}(E|Y_s) \leq \text{HN}(E|H) = m \cdot \text{HN}(E_{s_0})$ hold for all $s \in S$, where $Y_{s_0} = H$ is chosen so as to satisfy the Mehta-Ramanathan theorem (Proposition 4.5). As by hypothesis $\text{HN}(E_s) = \tau$ (constant) for all $s \in S$, it follows that $\text{HN}(E|Y_s) = m \cdot \tau$ for all $s \in S$. Moreover, by Proposition 4.4, the restriction of the canonical reduction of E_s to Y_s gives the canonical reduction of $E_s|Y_s$ for each $s \in S$. Hence if $(L, f : L \rightarrow E(V_P))$ and $(L', f' : L' \rightarrow E(V_P))$ are two relative canonical reductions for E , then $(L|Y, f|Y)$ and $(L'|Y, f'|Y)$ are two relative canonical reductions for $E|Y$. Hence by the inductive hypothesis, we must have an isomorphism $L|Y \cong L'|Y$ of line bundles on Y .

By Proposition 6.2, the restriction morphism of the relative Picard schemes

$$r : \text{Pic}_{X/S} \rightarrow \text{Pic}_{Y/S}$$

is a closed embedding when m is sufficiently large. Note that L and L' define global sections $[L]$ and $[L']$ of $\text{Pic}_{X/S} \rightarrow S$, which map under the restriction morphism $r : \text{Pic}_{X/S} \rightarrow \text{Pic}_{Y/S}$ to the same global section $[L|Y] = [L'|Y]$ of $\text{Pic}_{Y/S} \rightarrow S$. Hence $[L] = [L']$.

Recall that we have made an fppf base change such that $Y \rightarrow S$ (and hence also $X \rightarrow S$) admits a global section. Hence from $[L] = [L']$ we conclude that there exists a line bundle K on S and an isomorphism $L \otimes \pi^* K \rightarrow L'$. By shrinking S to a further smaller open neighbourhood of s_0 , we can assume that K is trivial on S , and hence we have an isomorphism $\psi : L \xrightarrow{\sim} L'$ on X . Hence we can replace (L', f') by $(L, f' \circ \psi)$ as the second of the two relative canonical reductions of E which we want to show to be isomorphic, that is, we are now reduced to the case where $L = L'$, and we have been given two relative canonical reductions (L, f) and (L, f') of E with $[L|Y, f|Y] = [L|Y, f'|Y]$. Note that by the uniqueness of canonical reduction over Y , there exists an automorphism $g : L|Y \rightarrow L|Y$ such that $f|Y = (f'|Y) \circ g$. Note that any automorphism g of a line bundle on Y is given by scalar multiplication by an invertible function $\gamma \in \Gamma(Y, \mathcal{O}_Y)^\times$. Hence we have $f|Y = \gamma \cdot (f'|Y)$.

As $\pi_Y = \pi|Y : Y \rightarrow S$ is smooth projective with geometrically connected fibers, in particular π_Y is proper flat with geometrically integral fibers. Hence by Lemma 7.2, the homomorphism $\pi_Y^\# : \mathcal{O}_S \rightarrow (\pi_Y)_* \mathcal{O}_Y$ is an isomorphism of sheaves of rings. It follows that any invertible function $\gamma \in \Gamma(Y, \mathcal{O}_Y)^\times$ is of the form $h \circ \pi_Y$ where $h = \gamma \circ b \in \Gamma(S, \mathcal{O}_S)^\times$ where $b : S \rightarrow Y$ is a global section.

By Theorem 7.1, if m is sufficiently large, then as the relative dimension n of X/S is ≥ 1 , we have

$$\pi_*(\underline{\text{Hom}}(L, E(V_P)(-m))) = 0.$$

By tensoring with $\underline{\text{Hom}}(L, E(V_P))$, the short exact sequence $0 \rightarrow \mathcal{O}_X(-m) \rightarrow \mathcal{O}_X \rightarrow \mathcal{O}_Y \rightarrow 0$ gives the short exact sequence

$$0 \rightarrow \underline{\text{Hom}}(L, E(V_P))(-m) \rightarrow \underline{\text{Hom}}(L, E(V_P)) \rightarrow \underline{\text{Hom}}(L|Y, (E|Y)(V_P)) \rightarrow 0.$$

The associated long exact cohomology sequence shows that the restriction homomorphism

$$\rho : H^0(X, \underline{\text{Hom}}(L, E(V_P))) \rightarrow H^0(Y, \underline{\text{Hom}}(L|Y, (E|Y)(V_P)))$$

is injective. As

$$\rho(f) = f|Y = h \cdot (f'|Y) = h \cdot \rho(f') = \rho(h \cdot f')$$

we conclude that $f = h \cdot f'$. Hence scalar multiplication by the element $h \in \Gamma(S, \mathcal{O}_S)^\times$ gives an isomorphism $\phi : L \rightarrow L$ such that $f = f' \circ \phi : L \rightarrow E(V_P)$, as desired. This completes the proof of Proposition 7.4.(2). \square

Remark 7.5 Let F be a vector bundle on a scheme X , let $Z \subset \mathbf{P}(F)$ be a closed subscheme, and let $f \in \Gamma(X, F)$ be a global section of F . Let $\pi : \mathbf{F} =$

$\mathrm{Spec}_{\mathcal{O}_X} \mathrm{Sym}_{\mathcal{O}_X}(F^\vee) \rightarrow X$ be the geometric vector bundle over X associated to F , and let $f : X \rightarrow \mathbf{F}$ denote its section defined by f . Let $\widehat{Z} \subset \mathbf{F}$ denote the affine cone over Z . Let $W = f^{-1}(\widehat{Z}) \subset X$ be its schematic inverse image under the section f . This has the property that if $\phi : X' \rightarrow X$ is any morphism of schemes, if $F' = \phi^*(F)$, if $Z' \subset \mathbf{P}(F')$ is the pullback of Z and if $f' \in \Gamma(X', F')$ is the pullback of f , then f' factors via \widehat{Z}' if and only if f factors via $W \hookrightarrow X$.

Proposition 7.6 (Lifting the relative canonical reduction) *Let $X \rightarrow S$ be a smooth projective morphism of noetherian k -schemes with geometrically connected fibers of dimension ≥ 2 , together with a given relatively very ample line bundle $\mathcal{O}_{X/S}(1)$ on X/S . Let E be a principal G -bundle on X . Let τ be the global maximum of $\mathrm{HN}(E_s)$ over S . Let $Y \subset X$ be a smooth relative effective divisor over S , such that $\mathcal{O}_X(Y) \cong \mathcal{O}_{X/S}(m)$, where m is sufficiently large. Let there exist a relative canonical reduction for the restriction $E|_Y$ over S , with constant HN-type $m\tau$. Then there exists a unique closed subscheme $S' \subset S$ with the universal property that for any morphism $T \rightarrow S$ where T is noetherian, the pull-back E_T admits a relative canonical reduction of type τ if and only if $T \rightarrow S$ factors via S' . Equivalently, we have*

- (i) *there exists a relative canonical reduction for $E_{S'}$ over S' of type τ , and*
- (ii) *if $T \rightarrow S$ is any noetherian base change such that E_T admits a relative canonical reduction of type τ , then $T \rightarrow S$ factors via S' .*

Proof. By uniqueness of relative canonical reductions (Proposition 7.4.(2)), a relative canonical reduction defined on an fppf (or étale) cover of S will descend to S , so it is enough to prove the result locally over an fppf (or étale) cover of S . In particular, we can assume that S is connected. Moreover, as explained in the course of the proof of Proposition 7.4, we can assume that Y has a global section over S .

By Proposition 6.2, the restriction morphism of relative Picard schemes $r : \mathrm{Pic}_{X/S} \rightarrow \mathrm{Pic}_{Y/S}$ is a closed embedding if $m \gg 0$. So if (L_Y, f_Y) is a relative canonical reduction for $E|_Y$, there exists a closed subscheme S_1 of S on which $[L_Y]$ lifts to a section of $\mathrm{Pic}_{X/S}$ and such that S_1 has the universal property described in Remark 6.4. Let $X_1 = X \times_S S_1$, $Y_1 = Y \times_S S_1$, and E_1 be the pullback of E to X_1 . As Y has a global section, so does Y_1 . As Y_1 (and hence X_1) has a global section, this means $L_Y|_{Y_1}$ lifts to a line bundle L_1 over X_1 , so we can assume $L_Y|_{Y_1} = L_1|_{Y_1}$.

For any E_s on X_s with $\mathrm{HN}(E_s) = \tau$, let d denote the degree of the line bundle L_s for any canonical reduction (L_s, f_s) of E_s . Then d is constant as S is connected (in fact, $d = \langle \lambda_P, \tau \rangle$). Let \mathbb{L} denote the relative Picard line bundle on $X \times \mathrm{Pic}_{X/S}^d$, normalized to be trivial on a chosen global section of $X \rightarrow S$. Consider the projection $p_2 : X \times \mathrm{Pic}_{X/S}^d \rightarrow \mathrm{Pic}_{X/S}^d$ and the vector bundle $F = \underline{\mathrm{Hom}}(\mathbb{L}, p_1^* E(V_P))$ on $X \times \mathrm{Pic}_{X/S}^d$. Applying Theorem 7.1 to the above data gives an integer m_0 with the property given in the conclusion of Theorem 7.1 for base changes $T \rightarrow \mathrm{Pic}_{X/S}^d$. We base change to $T = S_1$, under the morphism $[L_1] : S_1 \rightarrow \mathrm{Pic}_{X/S}^d$. Note that the pull-back of $p_2 : X \times \mathrm{Pic}_{X/S}^d \rightarrow \mathrm{Pic}_{X/S}^d$ is $\pi_1 : X_1 \rightarrow S_1$, and Zariski locally over S_1 , the pull-back of F is isomorphic to $\underline{\mathrm{Hom}}(L_1, E_1(V_P))$. Hence by Theorem 7.1

$R^{n-i}(\pi_1)_* \underline{Hom}(L_1, E_1(V_P)(-m)) = 0$ for $m \geq m_0$ and $i \geq 1$. As $n \geq 2$ by inductive hypothesis, both the 0th and 1st direct images vanish, and hence the restriction map

$$\rho : H^0(X_1, \underline{Hom}(L_1, E_1(V_P))) \rightarrow H^0(Y_1, \underline{Hom}(L_1|Y_1, (E_1|Y_1)(V_P)))$$

is a bijection for $m \geq m_0$. Hence when m is sufficiently large, there exists a unique $f_1 : L_1 \rightarrow E_1(V_P)$ such that $f_Y|Y_1 = f_1|Y_1$. This gives a pair $(L_1, f_1 : L_1 \rightarrow E_1(V_P))$ which lifts (that is, prolongs) the pair (L_Y, f_Y) from Y_1 to X_1 . However, (L_1, f_1) need not be a relative canonical reduction over S_1 . We will produce a maximal closed subscheme $S' \subset S_1$ over which it will be so.

By tensoring with L_1^{-1} , the homomorphism $f_1 : L_1 \rightarrow E_1(V_P)$ can be regarded as a section $f_1 \in \Gamma(X_1, E_1(V_P) \otimes L_1^{-1})$. Note that we have a natural identification $\mathbf{P}(E_1(V_P)) = \mathbf{P}(E_1(V_P) \otimes L_1^{-1})$ under which E_1/P becomes a closed subscheme $E_1/P \subset \mathbf{P}(E_1(V_P) \otimes L_1^{-1})$. Let $W = f_1^{-1}(\widehat{E_1/P}) \subset X_1$ be the closed subscheme of X_1 defined as in Remark 7.5.

Consider the composite $W \hookrightarrow X_1 \rightarrow S_1$, which is a projective morphism. The flattening stratification of S_1 for this morphism has strata $S_{1,p(t)}$ indexed by the various Hilbert polynomials $p(t) \in \mathbb{Q}[t]$ of the fibers w.r.t. $\mathcal{O}_{X_1/S_1}(1)$. The top stratum $S' = S_{1,p_0(t)}$, with Hilbert polynomial $p_0(t)$ equal to the Hilbert polynomial of the fibers of $X \rightarrow S$, is a closed subscheme of S_1 . We will show that S' satisfies the proposition.

Proof that S' satisfies 7.6.(i): To begin with, we will look at the points of S' . Let (L', f') denote the base-change of (L_1, f_1) under $S' \hookrightarrow S_1$. By definition, $s \in S'$ if and only if the fiber of $W \rightarrow S$ over s is X_s , which means the section $f'_s \in \Gamma(X_s, E_s(V_P) \otimes L_s^{-1})$ lies in the affine cone W_s over $E_s/P \subset \mathbf{P}(E_s(V_P) \otimes L_s^{-1})$. Let $U \subset X_s$ be the open subscheme where $\text{rank}(f'_x) = 1$. Then U is non-empty and big in X_s , as by assumption for each $s \in S'$, the intersection $U \cap Y_s$ is a big open subscheme of Y_s . Hence $[L'_s, f'_s] \in R(E_s/X_s, P)$, that is, (L'_s, f'_s) is a linearized rational reduction of E_s .

Let $\mu_{(P,\sigma)}(E_s)$ and $\mu_{(P,\sigma|(U \cap Y_s))}(E_s|Y_s)$ be the types of the rational P -reductions $\sigma : U \rightarrow E_s/P$ and $\sigma|(U \cap Y_s) : U \cap Y_s \rightarrow (E_s|Y_s)/P$ respectively. As U and $U \cap Y_s$ are both big, by Lemma 4.2 we get the equality

$$m \cdot \mu_{(P,\sigma)}(E_s) = \mu_{(P,\sigma|(U \cap Y_s))}(E_s|Y_s).$$

As by assumption $\mu_{(P,\sigma|(U \cap Y_s))}(E_s|Y_s) = \text{HN}(E_s|Y_s) = m \cdot \tau$, it follows that $\mu_{(P,\sigma)}(E_s) = \tau$. Now by hypothesis of Proposition 7.6, τ is the global maximum of $\text{HN}(E_s)$ over S . Hence by the maximality property of canonical reductions (see Statement 4.1), $\sigma : U \rightarrow E_s/P$ is the canonical reduction of E_s . Hence we have shown that (L'_s, f'_s) defines the canonical reduction of E_s at all points $s \in S'$.

Let $X' = X \times_S S'$, $Y' = Y \times_S S'$ and $W' = W \times_S S'$. Note that by definition of S' as the top stratum in the flattening stratification of $W \rightarrow S$ with Hilbert polynomial that of fibers of $X \rightarrow S$, we see that $W' \subset X'$ is a closed subscheme, and both $W' \rightarrow$

S' and $X' \rightarrow S'$ are flat projective with the same Hilbert polynomial p_0 of fibers. Hence we must have $W' = X'$. This shows that as schemes, $X' \subset W \subset X$. Let $E' = E|X'$. As above, $f' : L' \rightarrow E'(V_P)$ denote the restriction of $f_1 : L_1 \rightarrow E_1(V_P)$ to X' . As shown above, for each $s \in S'$ the restriction $f'_s = f'|X_s$ has rank 1 over a big open subscheme of X_s , which shows that $f' : L' \rightarrow E'(V_P)$ has rank 1 over a relatively big open subscheme of X' over S' . As $X' \subset W$, by Remark 7.5, the section of $\mathbf{P}(E'(V_P))$ defined by (L', f') over the above relatively big open subscheme factors via $E'/P \hookrightarrow \mathbf{P}(E'(V_P))$. Hence (L', f') is a linearized relative rational P -reduction of E' over S' in the sense of Definition 3.5. Now recall that we have shown that for each $s \in S'$, (L', f') restricts to X_s to give the canonical reduction of E_s . Hence by Definition 5.1, (L', f') is a relative canonical of E' over S' . This completes the proof of the property (i) in the conclusion of Proposition 7.6.

Proof that S' satisfies 7.6.(ii): Let $\phi : T \rightarrow S$ be a morphism, and let (\mathbb{L}, g) be a relative canonical reduction of E_T on X_T/T . Note that the pull-back of the given relative canonical reduction (L_Y, f_Y) of $E|Y$ under $\phi : T \rightarrow S$ is a relative canonical reduction of $E_T|Y_T$ over T . Therefore by Propositions 5.2 and 7.4 applied over T , the restriction of (\mathbb{L}, g) to Y_T is isomorphic to (ϕ^*L_Y, ϕ^*f_Y) .

In particular, as \mathbb{L} is a prolongation of $\phi^*(L_Y)$ under $Y_T \subset X_T$, by Remark 6.4, the morphism $\phi : T \rightarrow S$ factors via $S_1 \hookrightarrow S$, giving $\phi_1 : T \rightarrow S_1$. The pullback of $f_1 : L_1 \rightarrow E_1(V_P)$ under $\phi_1 : T \rightarrow S_1$ gives another prolongation $(\phi_1^*L_1, \phi_1^*f_1)$ of (ϕ^*L_Y, ϕ^*f_Y) under $Y_T \subset X_T$. Zariski locally over T , we can identify \mathbb{L} with $\phi_1^*L_1$ because of Proposition 6.2. By Theorem 7.1, the restriction map

$$\rho : H^0(X_T, \underline{\text{Hom}}(\mathbb{L}, E_T(V_P))) \rightarrow H^0(Y_T, \underline{\text{Hom}}(\mathbb{L}|Y_T, (E_T|Y_T)(V_P)))$$

is injective, hence we must moreover have $(\phi_1^*L_1, \phi_1^*f_1) \cong (\mathbb{L}, g)$, that is, there exists an isomorphism $u : \phi_1^*L_1 \rightarrow \mathbb{L}$ such that $\phi_1^*f_1 = g \circ u : \phi_1^*L_1 \rightarrow E_T(V_P)$.

As (\mathbb{L}, g) is a relative canonical reduction, it follows that so is $(\phi_1^*L_1, \phi_1^*f_1)$. Hence $\phi_1^*f_1$ factors via the cone $\widehat{E_T/P}$. Therefore by Remark 7.5, the morphism $(\phi_1)_{X_1} : X_T \rightarrow X_1$ must factor through $W \hookrightarrow X_1$. As the inclusion $W \hookrightarrow X_1$ is monic, it follows that $X_T \rightarrow T$ is the base-change of $W \rightarrow S_1$ under ϕ_1 . As $X_T \rightarrow T$ is flat with Hilbert polynomial $p_0(t)$, by the universal property of the flattening stratification for $W \rightarrow S_1$, the morphism $\phi : T \rightarrow S_1$ factors via the corresponding stratum $S' = S_{1, p_0(t)}$ as we wished to show. This completes the proof of Proposition 7.6. \square

Proof of Theorem 1.2. With all the necessary ingredients in place at last, we can now complete the proof of Theorem 1.2. We proceed by induction on the relative dimension of X/S . We have earlier proved the theorem when X/S is of relative dimension 1 (see [Gu-Ni 1]). So now assume that the relative dimension is ≥ 2 .

It follows from the uniqueness of a relative canonical reduction proved in Proposition 7.4.(2) that the assertion of the Theorem 1.2 is local over the base S , that is, it is enough to prove it in a neighbourhood of each point s_0 of S . As by Proposition 7.4.(1) the HN-type $\text{HN}(E_s)$ is upper semicontinuous on S , by shrinking the neighbourhood

of s_0 if needed, we can assume that $\text{HN}(E_s)$ attains a unique maximum value τ at s_0 . Hence $|S|^\tau(E)$ is a closed subset of $|S|$. By the Mehta-Ramanathan theorem, given any sufficiently large $m \in \mathbb{Z}$, there will exist an effective divisor $H \subset X_{s_0}$ which is smooth over s_0 with $H \in |\mathcal{O}_{X_{s_0}}(m)|$, such that the canonical reduction of E_{s_0} on restriction to H gives the canonical reduction of $E_{s_0}|_H$. This includes the property that the domain of definition of the canonical reduction of E_{s_0} , which is a big open subset of X_{s_0} , intersects H to give a big open subset of H .

If m is taken to be sufficiently large, then after shrinking S to a smaller neighbourhood of s_0 if needed, there will exist an effective relative divisor $Y \subset X$ over S which is smooth over S with $Y \in |\mathcal{O}_{X/S}(m)|$, such that $Y_{s_0} = H$. In particular, we have $\text{HN}(E|_{Y_{s_0}}) = m\tau$. Now by shrinking S further, we can assume that for the restricted family $E|_Y$ on Y over S , $m\tau$ is the unique maximum value of $\text{HN}(E|_Y)$ on S .

By our inductive hypothesis on the relative dimension of $X \rightarrow S$, the Theorem 1.2 holds for the principal bundle $E|_Y$ on Y over S . Hence there exists a closed subscheme $S^{m\tau}(E|_Y) \subset S$ with the desired universal property for $E|_Y$ for the type $m\tau$. For any $s \in |S|^\tau(E) \subset |S|$ we have $\text{HN}(E|_{Y_s}) \leq m\tau$ as $m\tau$ is the maximum for $\text{HN}(E|_{Y_s})$ by assumption. On the other hand, by Proposition 4.4, if $s \in |S|^\tau(E)$ then the restriction of canonical reduction of E_s to the subscheme $Y_s \subset X_s$ prolongs to a big open subset of Y_s to give us a rational parabolic reduction of $E|_{Y_s}$ of a type which is $\geq m\tau$. It follows that $\text{HN}(E|_{Y_s}) = m\tau$ whenever $s \in |S|^\tau(E) \subset |S|$ provided that τ is the unique maximum value of $\text{HN}(E_s)$ and $m\tau$ is the unique maximum value of $\text{HN}(E|_Y)$ on S . Hence at the level of sets, we have the inclusion $|S|^\tau(E) \subset |S|^{m\tau}(E|_Y)$, when S is replaced by a neighbourhood of s_0 in S .

By Proposition 5.2, with S and τ as above, if $T \rightarrow S$ is a base change such that E_T admits a relative canonical reduction of constant type τ , then this reduction restricts to Y_T to give a relative canonical reduction of $E_T|_{Y_T}$ of type $m\tau$. Hence $T \rightarrow S$ must factor through $S^{m\tau}(E|_Y) \hookrightarrow S$, so if the theorem is true, then $S^\tau(E)$ must be a closed subscheme of $S^{m\tau}(E|_Y)$.

It therefore only remains to identify $S^\tau(E)$ as an appropriate closed subscheme of $S^{m\tau}(E|_Y)$, and to show that it has the desired universal property. So we can replace the original S by $S^{m\tau}(E|_Y)$, and assume (by the inductive hypothesis) that $E|_Y$ has a relative canonical reduction over S . Moreover, we can assume (by replacing S by an open neighbourhood of s_0 if necessary) that τ is the global maximum for $\text{HN}(E_s)$ over s . The problem is to lift (prolong) this rational reduction of the structure group from $E|_Y$ to E . By Proposition 7.6, there exists a unique largest closed subscheme $S' \subset S^{m\tau}(E|_Y)$ over which such a lift exists, and this closed scheme S' has the functorial property which we would like $S^\tau(E)$ to possess. This completes the proof of Theorem 1.2. \square

The Theorem 1.1 can now be easily deduced from Theorem 1.2 as follows. Recall that the stack $\text{Bun}_{X/S}(G)$ of principal G -bundles on X/S is the stack over S whose objects over an S -scheme T are all principal G -bundle E on X_T . That $\text{Bun}_{X/S}(G)$

is an Artin stack can be seen as follows. The group G can be embedded as a closed subgroup in $GL_{n,k}$ for some n , which gives a 1-morphism $Bun_{X/S}(G) \rightarrow Bun_{X/S}(GL_{n,k})$. By using an appropriate Hilbert scheme of sections, it can be seen that the 1-morphism $Bun_{X/S}(G) \rightarrow Bun_{X/S}(GL_{n,k})$ is representable, in fact, schematic.

For each $\tau \in \overline{C}$, we define an S -stack $Bun_{X/S}^\tau(G)$ whose objects over an S -scheme T are all triples (E, L, f) consisting of a principal G -bundle E on X_T , together with a relative canonical reduction (L, f) of type τ . Forgetting the reduction (L, f) gives a 1-morphism $i_\tau : Bun_{X/S}^\tau(G) \rightarrow Bun_{X/S}(G)$.

It can be directly seen that the above morphism is representable and so $Bun_{X/S}^\tau(G)$ is an Artin stack, even when the characteristic of k is arbitrary and when the Behrend conjecture does not hold (see [Gu-Ni 3]). In our present context, the Theorem 1.2, together with Proposition 7.4, show that the 1-morphism $i_\tau : Bun_{X/S}^\tau(G) \rightarrow Bun_{X/S}(G)$ is a locally closed embedding of stacks (see [La-MB] Definition 3.14) (in particular, this gives another proof that $Bun_{X/S}^\tau(G)$ is algebraic). This completes the proof of Theorem 1.1.

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